DE

AEQUATIONIBUS DIFFERENTIALIBUS,

QUIBUS AEQUILIBRIUM ET MOTUS

LAMINAE CRYSTALLINAE DEFINIUNTUR.

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QUAM

CONSENSU ET AUCTORITATE

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§. 1.

In problemate aequilibrii et motus laminae elasticae tractando cl. Sophie Germain, Lagrange, Poisson conjecturas fecerunt, quas falsas esse cl. Kirchhoff in diarii Crelliani tomo XL. demonstravit. Qui novam theoriam deduxit ex principiis, quae quidem non minus sunt hypothetica, quae tamen similia sunt iis, quibus cl. Jac. Bernoulli usus est ad baculi elastici aequilibrium et oscillationes definienda et quae theoriam praebuerunt satis experimentis congruentem. In recentiore commentatione (Diar. Crell. LVI) cl. Kirchhoff theorema proposuit generale, cujus ope fere omnia elasticitatis problemata accuratissime solvi possunt. Unde mihi liceat, nulla facta conjectura physica, solum mathematicis deliberationibus utens, aequationes deducere, quarum integratione aequilibrium et motus laminae elasticae crystallinae vel non crystallinae definiuntur sub ea conditione, ut tantum infinite paullum ex aequilibrii statu lamina progrediatur. Quae theoria a me constituetur, in casu laminae non crystallinae easdem praebet aequationes ac inventas a cl. Kirchhoff et confirmat igitur conjecturas, quibus usus est. Sed non minus facile casus laminae crystallinae ea continetur.

§. 2.

Theorema cl. Kirchhoff hoc est.

Punctum corporis elastici habeat in statu naturali coordinatas x, y, z, forma corporis mutata per quascunque vires, idem punctum habeat coordinatas $x + u$, $y + v$, $z + w$. Designentur per

$$x_x = \frac{du}{dx}, \qquad y_z = z_y = \frac{dw}{dy} + \frac{dv}{dz}.$$

$$y_y = \frac{dv}{dy}, \qquad z_x = x_z = \frac{du}{dz} + \frac{dw}{dx}.$$

$$z_z = \frac{dw}{dz}, \qquad x_y = y_x = \frac{dv}{dx} + \frac{du}{dy}.$$

Formetur functio F homogenea secundi gradus sex variabilium x_x, y_y, z_z, y_z, z_x, x_y, cujus coefficientes constantes sint, a natura corporis dependentes; sit porro $\delta\,\Omega$ momentum externarum virium corpus afficientium; quibus constitutis, erit

$$(1) \qquad \delta\,\Omega - \delta \int F\,dx\,dy\,dz = 0.$$

ubi $\int$ extendendum est ad totum corpus.

Quae aequatio valet, si omnes dimensiones corporis ejusdem sunt ordinis. Cui conditioni quum corpus non satisfaciat, fingatur in partes divisum esse corpus, quarum dimensiones omnes aequales inter se sint. Tum aequatio (1) pro quaque parte proponatur et summa omnium formetur; itaque fiet:

$$(2) \qquad \delta\,\Omega - \delta\,\Sigma \int F\,dx\,dy\,dz = 0,$$

ubi $\delta\,\Omega$ momentum virium externarum significat, quae et interiorem partem et superficiem corporis afficiunt, quum momentum virium elasticarum, quae superficiem singulas partes limitantem afficiunt, evanescat.

Qua ab aequatione aequilibrium corporis elastici definiente, ope noti dynamices principii, ad motus casum transibimus.

Erit

$$(3) \qquad \int dt\,(\delta\,T + \delta\,\Omega - \delta\,\Sigma \int F\,dx\,dy\,dz) = 0,$$

ubi t tempus, T dimidiam vim vivam significant.

Quibus aequationibus (2), (3) utar ad aequilibrium et oscillationes laminae cujusvis naturae definienda.

§. 3.

In area media laminae definiatur punctum P ita, ut coordinatae relatae ad systema orthogonale in area media firmum sint a et b; per quod punctum quum ponantur tria elementa linearia 0, 1, 2, ita ut 0 sit normalis areae mediae 1, 2 in area media sint inter se perpendicularia, parallela coordinatarum axibus a et b. Forma laminae mutata haec elementa universo non amplius inter se erunt perpendicularia, sed angulos formabunt, quorum cosinus a magnitudine dilatationum vel contractionum factarum dependebunt. Situs puncti alicujus vicini puncto P referatur in systema orthogonale, cujus initium sit P, cujus axis z coincidat directioni elementi 0, cujus axes x et y sint paralleli resp. a et b, coincidant itaque in statu naturali cum elementis 1 et 2. Cujus puncti sint coordinatae x, y, z. Forma autem laminae mutata coordinatae ejus erunt $x + u$, $y + v$, $z + w$. Erunt itaque cosinus angulorum, quae forma laminae mutata sunt inter 0, 1, 2 et coordinatarum axes

$$\begin{matrix} 1, & \frac{dv}{dx}, & \frac{dw}{dx} \\ \frac{du}{dy}, & 1, & \frac{dw}{dy} \\ \frac{du}{dz}, & \frac{dv}{dz}, & 1 \end{matrix}$$

Porro coordinatae puncti P referantur ad systema quodvis in spatio situm, cujus axes cum $x\,y\,z$ angulos forment, quorum cosinus sint

$$\begin{array}{l} \alpha_0,\ \beta_0,\ \gamma_0, \\ \alpha_1,\ \beta_1,\ \gamma_1, \\ \alpha_2,\ \beta_2,\ \gamma_2, \end{array}$$

ita, ut index 0 attineat ad axem x, indices 1, 2 ad axes resp. y, z. Quo in coordinatarum systemate $x+u$, $y+v$, $z+w$ erunt, si initium ejus habet cordinatas $\xi\,\eta\,\zeta$

$$(4)\quad \begin{cases} \xi + \alpha_0\,(x+u) + \alpha_1\,(y+v) + \alpha_2\,(z+w). \\ \eta + \beta_0\,(x+u) + \beta_1\,(y+v) + \beta_2\,(z+w). \\ \zeta + \gamma_0\,(x+u) + \gamma_1\,(y+v) + \gamma_2\,(z+w). \end{cases}$$

Quas quantitates necesse est, esse functiones quantitatum $a+x$, $b+y$ i. e. quoti differentiales earum secundum a et b respective differentialibus secundum x et y aequales esse debent.

Quare fiunt hae aequationes:

$$(5\text{a})\quad \begin{cases} \alpha_0\,(1+\frac{du}{dx}) + \alpha_1\,\frac{dv}{dx} + \alpha_2\,\frac{dw}{dx} = \frac{d\xi}{da} + \frac{d\alpha_0}{da}\,(x+u) + \frac{d\alpha_1}{da}\,(y+v) + \frac{d\alpha_2}{da}\,(z+w) \\ \qquad + \alpha_0\,\frac{du}{da} + \alpha_1\,\frac{dv}{da} + \alpha_2\,\frac{dw}{da}. \\ \beta_0\,(1+\frac{du}{dx}) + \beta_1\,\frac{dv}{dx} + \beta_2\,\frac{dw}{dx} = \frac{d\eta}{da} + \frac{d\beta_0}{da}\,(x+u) + \frac{d\beta_1}{da}\,(y+v) + \frac{d\beta_2}{da}\,(z+w) \\ \qquad + \beta_0\,\frac{du}{da} + \beta_1\,\frac{dv}{da} + \beta_2\,\frac{dw}{da}. \\ \gamma_0\,(1+\frac{du}{dx}) + \gamma_1\,\frac{dv}{dx} + \gamma_2\,\frac{dw}{dx} = \frac{d\zeta}{da} + \frac{d\gamma_0}{da}\,(x+u) + \frac{d\gamma_1}{da}\,(y+v) + \frac{d\gamma_2}{da}\,(z+w) \\ \qquad + \gamma_0\,\frac{du}{da} + \gamma_1\,\frac{dv}{da} + \gamma_2\,\frac{dw}{da}. \end{cases}$$

$$(5\text{b})\quad \begin{cases} \alpha_0\,\frac{du}{dy} + \alpha_1\,(1+\frac{dv}{dy}) + \alpha_2\,\frac{dw}{dy} = \frac{d\xi}{db} + \frac{d\alpha_0}{db}\,(x+u) + \frac{d\alpha_1}{db}\,(y+v) + \frac{d\alpha_2}{db}\,(z+w) \\ \qquad + \alpha_0\,\frac{du}{db} + \alpha_1\,\frac{dv}{db} + \alpha_2\,\frac{dw}{db}. \\ \beta_0\,\frac{du}{dy} + \beta_1\,(1+\frac{dv}{dy}) + \beta_2\,\frac{dw}{dy} = \frac{d\eta}{db} + \frac{d\beta_0}{db}\,(x+u) + \frac{d\beta_1}{db}\,(y+v) + \frac{d\beta_2}{db}\,(z+w) \\ \qquad + \beta_0\,\frac{du}{db} + \beta_1\,\frac{dv}{db} + \beta_2\,\frac{dw}{db}. \\ \gamma_0\,\frac{du}{dy} + \gamma_1\,(1+\frac{dv}{dy}) + \gamma_2\,\frac{dw}{dy} = \frac{d\zeta}{db} + \frac{d\gamma_0}{db}\,(x+u) + \frac{d\gamma_1}{db}\,(y+v) + \frac{d\gamma_2}{db}\,(z+w) \\ \qquad + \gamma_0\,\frac{du}{db} + \gamma_1\,\frac{dv}{db} + \gamma_2\,\frac{dw}{db}. \end{cases}$$

Cosinus angulorum, qui sunt mutata laminae forma inter elementa 1, 2 et axes novos in ratione sunt:

$$cos\,(1,\xi) : cos\,(1,\eta) : cos\,(1,\zeta) = \frac{d\xi}{da} : \frac{d\eta}{da} : \frac{d\zeta}{da}.$$

$$cos\,(2,\xi) : cos\,(2,\eta) : cos\,(2,\zeta) = \frac{d\xi}{db} : \frac{d\eta}{db} : \frac{d\zeta}{db}.$$

Itaque posito

$$\text{(6a)} \qquad \sqrt{\left(\frac{d\xi}{da}\right)^2 + \left(\frac{d\eta}{da}\right)^2 + \left(\frac{d\zeta}{da}\right)^2} = 1 + \varepsilon$$

$$\text{(6b)} \qquad \sqrt{\left(\frac{d\xi}{db}\right)^2 + \left(\frac{d\eta}{db}\right)^2 + \left(\frac{d\zeta}{db}\right)^2} = 1 + \varepsilon_1.$$

fiunt

$$\frac{d\xi}{da} = cos\,(1,\xi)\,(1+\varepsilon), \qquad \frac{d\xi}{db} = cos\,(1,\xi)\,(1+\varepsilon_1)$$

$$\frac{d\eta}{da} = cos\,(1,\eta)\,(1+\varepsilon), \qquad \frac{d\eta}{db} = cos\,(1,\eta)\,(1+\varepsilon_1)$$

$$\frac{d\zeta}{da} = cos\,(1,\zeta)\,(1+\varepsilon), \qquad \frac{d\zeta}{db} = cos\,(1,\zeta)\,(1+\varepsilon_1).$$

Quum tamen sit,

$$cos\,(1,\xi) = cos\,(1,x)\;cos\,(x,\xi) + cos\,(1,y)\;cos\,(y,\xi) + cos\,(1,z)\;cos\,(z,\xi).$$

etc. invenitur

$$cos\,(1,\xi) = \alpha_0 + \alpha_1 \frac{dv}{dx} + \alpha_2 \frac{dw}{dx}, \qquad cos\,(2,\xi) = \alpha_0 \frac{du}{dy} + \alpha_1 + \alpha_2 \frac{dw}{dy}$$

$$cos\,(1,\eta) = \beta_0 + \beta_1 \frac{dv}{dx} + \beta_2 \frac{dw}{dx}, \qquad cos\,(2,\eta) = \beta_0 \frac{du}{dy} + \beta_1 + \beta_2 \frac{dw}{dy}$$

$$cos\,(1,\zeta) = \gamma_0 + \gamma_1 \frac{dv}{dx} + \gamma_2 \frac{dw}{dx}, \qquad cos\,(2,\zeta) = \gamma_0 \frac{du}{dy} + \gamma_1 + \gamma_2 \frac{dw}{dy}.$$

Qui valores tamen hac deliberatione multo simpliciores fiunt.

Ex aequatione (3) inveniuntur valores quantitatum x_x y_y z_z etc., ex his valores quantitatum u v w. Quum respiciamus definitiones quantitatum x_x y_y z_z, facile intelligimus ad u v w addendas esse quantitates resp.

$$a_0 + cy - bz,\; b_0 + az - cx,\; c_0 + bx - ay$$

in quibus a_0, b_0, c_0, a, b, c sint constantes arbitrariae; quae constantes ita definiantur, ut in puncto $x=0$ $y=0$ $z=0$ sint:

$$u=0,\ v=0,\ w=0,\ \frac{dv}{dx}=0,\ \frac{dw}{dx}=0,\ \frac{dw}{dy}=0.$$

Quum elementa (1), (2) per punctum $x=0$, $y=0$, $z=0$ eant, aequationes conditionales constitutae exigunt, ut sit,

$$cos\,(1,\xi)=\alpha_0,\qquad cos\,(1,\eta)=\beta_0,\qquad cos\,(1,\zeta)=\gamma_0.$$

$$cos\,(2,\xi)=(\alpha_0\left(\frac{du}{dy}\right)_{x=0\,y=0\,z=0}+\alpha_1);\ cos\,(2,\eta)=(\beta_0\left(\frac{du}{dy}\right)_{x=0\,y=0\,z=0}+\beta_1);$$

$$cos\,(2,\zeta)=(\gamma_0\left(\frac{du}{dy}\right)_{x=0\,y=0\,z=0}+\gamma_1).$$

Posito igitur $\left(\frac{du}{dy}\right)_{x=0\,y=0\,z=0}=\sigma$, facile intelligitur

$$\frac{d\xi}{da}=\alpha_0\,(1+\varepsilon)\qquad \frac{d\xi}{db}=(\alpha_0\,\sigma+\alpha_1)\,(1+\varepsilon_1)$$

(8) $$\frac{d\eta}{da}=\beta_0\,(1+\varepsilon)\qquad \frac{d\eta}{db}=(\beta_0\,\sigma+\beta_1)\,(1+\varepsilon_1)$$

$$\frac{d\zeta}{da}=\gamma_0\,(1+\varepsilon)\qquad \frac{d\zeta}{db}=(\gamma_0\,\sigma+\gamma_1)\,(1+\varepsilon_1)\ \text{esse.}$$

Brevitatis causa introducantur quantitates

$$p\ =\alpha_1\frac{d\alpha_2}{da}+\beta_1\frac{d\beta_2}{da}+\gamma_1\frac{d\gamma_2}{da}$$

$$q\ =\alpha_2\frac{d\alpha_0}{da}+\beta_2\frac{d\beta_0}{da}+\gamma_2\frac{d\gamma}{da}$$

(9) $$r\ =\alpha_0\frac{d\alpha_1}{da}+\beta_0\frac{d\beta_1}{da}+\gamma_0\frac{d\gamma_1}{da}$$

$$p_1=\alpha_1\frac{d\alpha_2}{db}+\beta_1\frac{d\beta_2}{db}+\gamma_1\frac{d\gamma_2}{db}$$

$$q_1=\alpha_2\frac{d\alpha_0}{db}+\beta_2\frac{d\beta_0}{db}+\gamma_2\frac{d\gamma_0}{db}$$

$$r_1=\alpha_0\frac{d\alpha_1}{db}+\beta_0\frac{d\beta_1}{db}+\gamma_0\frac{d\gamma_1}{db}.$$

Quorum signornm ratione habita, aequationes (5a) (5b) multiplicentur resp. per α_0 β_0 γ_0 α_1 β_1 γ_1 α_2 β_2 γ_2 et addantur.

Quare fiunt, si respiciuntur sex relationes cognitae inter cosinus α_0 β_0 γ_0 etc.

$$\frac{du}{dx}=\frac{du}{da}+r(y+v)-q(z+w)+\varepsilon$$

$$\frac{dv}{dx}=\frac{dv}{da}+p(z+w)-r(x+u)$$

$$\frac{dw}{dx}=\frac{dw}{da}+q(x+u)-p(y+v)$$

$$\frac{du}{dy}=\frac{du}{db}+r_1(y+v)-q_1(z+w)+\sigma(1+\varepsilon_1)$$

$$\frac{dv}{dy}=\frac{dv}{db}+p_1(z+w)-r_1(x+u)+\varepsilon_1$$

$$\frac{dw}{dy}=\frac{dw}{db}+q_1(x+u)-p_1(y+v)$$

Quum $\frac{du}{da}$, $\frac{dv}{da}$, $\frac{dw}{da}$, $\frac{du}{db}$, $\frac{dv}{db}$, $\frac{dw}{db}$ non sint ∞ ad u, v, w, sunt ∞ parvi ad $\frac{du}{dx}$, $\frac{dv}{dx}$, $\frac{dw}{dx}$, $\frac{du}{dy}$, $\frac{dv}{dy}$, $\frac{dw}{dy}$, quod $u\,v\,w$ sunt ∞ parvi ad has quantitates (vide comment. cl. Kirchhoff Crell. LVI pag. 294), omissis itaque quantitatibus superiorum ordinum scribere licebit

$$(10)\qquad \begin{aligned} \frac{du}{dx}&=ry-qz+\varepsilon & \frac{du}{dy}&=r_1y-q_1z+\sigma\\ \frac{dv}{dx}&=pz-rx & \frac{dv}{dy}&=p_1z-r_1x+\varepsilon_1\\ \frac{dw}{dx}&=qx-py & \frac{dw}{dy}&=q_1x-p_1y. \end{aligned}$$

Differentiatae tres primae secundum y, sequentes secundum x relationes praebent:

$$\frac{d^2u}{dx\,dy}=r=\;\;0 \qquad \frac{d^2v}{dx\,dy}=0=-r_1 \qquad \frac{d^2w}{dx\,dy}=-p=q_1.$$

Quibus valoribus substitutis, fiunt aequationes:

$$\text{(10 a)}\qquad \begin{aligned} \frac{du}{dx} &= -qz+\varepsilon & \frac{du}{dy} &= pz+\sigma \\ \frac{dv}{dx} &= pz & \frac{dv}{dy} &= p_1 z+\varepsilon_1 \\ \frac{dw}{dx} &= qx-py & \frac{dw}{dy} &= -px-p_1 y. \end{aligned}$$

Ratione habita, quod quantitates $p, q, p_1, \varepsilon, \varepsilon_1, \sigma$ non sunt functiones coordinatarum $x\ y\ z$, nunc facile ad integrationem progredi prosumus, quae se sua sponte praebet; fit enim

$$\text{(11)}\qquad \begin{aligned} u &= z(-qx+py)+\varepsilon x+\sigma y+u_0 \\ v &= z(px+p_1 y)+\varepsilon_1 y+v_0 \\ w &= \frac{qx^2}{2}-pxy-\frac{p_1 y^2}{2}+w_0, \end{aligned}$$

ubi $u_0\ v_0\ w_0$ functiones sunt solius z et repraesentant valores quantitatum $u\ v\ w$ pro $x=0$ et $y=0$.

Etiam nunc fiunt:

$$\text{(12)}\qquad \begin{aligned} x_x &= \frac{du}{dx} = -qz+\varepsilon \\ y_y &= \frac{dv}{dy} = p_1 z+\varepsilon_1 \\ z_z &= \frac{dw}{dz} = \frac{dw_0}{dz} \\ y_z = z_y &= \frac{dw}{dy}+\frac{dv}{dz} = \frac{dv_0}{dz} \\ z_x = x_z &= \frac{du}{dz}+\frac{dw}{dx} = \frac{du_0}{dz} \\ x_y = y_x &= \frac{dv}{dx}+\frac{du}{dy} = 2pz+\sigma. \end{aligned}$$

Adnotamus, quantitates $x_x\ x_y$ etc. omnes esse functiones solius z.

Quum designentur quoti partiales differentiales functionis F secundum $x_x\ x_y$ etc. respective per $X_x\ X_y$ etc., aequationes generales elasticitatis fiunt (vid. Diar. Crell. LVI. pag. 290.):

$$\text{(13)}\qquad \frac{dX_x}{dx}+\frac{dX_y}{dy}+\frac{dX_z}{dz}=0$$

$$\frac{dY_x}{dx} + \frac{dY_y}{dy} + \frac{dY_z}{dz} = 0$$

$$\frac{dZ_x}{dx} + \frac{dZ_y}{dy} + \frac{dZ_z}{dz} = 0,$$

quae, quum respiciatur, quantitates x_x x_y etc. functiones solius z esse, hanc induunt formam

$$\frac{dX_z}{dz} = 0$$

$$(13a) \qquad \frac{dY_z}{dz} = 0$$

$$\frac{dZ_z}{dz} = 0$$

Ex quibus et ope conditionum (7) valores quantitatum u_0 v_0 w_0 definiuntur, itaque x_x, x_y etc. functiones cognitae solius z et quantitatum p q h_1 ε ε_1 σ fiunt, substituendae in functionem F. Quod jam supra adnotavimus functionis F coefficientes a natura laminae dependere, nunc nobis respiciendum est, itaque primum casum laminae non crystallinae nobis liceat inquirere.

§. 4.

In mediis non crystallinis functio F invenitur (vid. loc. cit.)

$$(14) \quad F = K \left\{ x_x^2 + y_y^2 + z_z^2 + \frac{1}{2} y_z^2 + \frac{1}{2} z_x^2 + \frac{1}{2} x_y^2 + \Theta (x_x + y_y + z_z)^2 \right\}$$

Tantum duae constantes in F inveniuntur, in casu generali 21, quod functio F 21 habet coefficientes, qui omnes inter se diversi esse possunt.

$$\frac{dF}{dx_z} = X_z \text{ fit igitur} = K x_z$$

$$\text{per analogiam } Y_z = K y_z$$

$$X_y = K x_y$$

$$Z_z = 2K(\Theta x_x + \Theta y_y + (1 + \Theta) z_z) \text{ Aequationes}$$

$$(13a) \text{ dicunt } \frac{dX_z}{dz} = 0 \text{ i. e. } K \frac{d^2 u_0}{dz^2} = 0.$$

$$\frac{dY_z}{dz} = 0 \text{ i. e. } K \frac{d^2 v_0}{dz^2} = 0.$$

Integratione invenitur

$$u_0 = \lambda_0 z + \mu_0$$

$$v_0 = \lambda_1 z + \mu_1$$

Conditioni, ut pro $z = 0$ u_0 et v_0 evanescant, satisfacere poterimus ponendo $\mu_0 = 0$ $\mu_1 = 0$. λ_0 λ_1 sunt constantes arbitrariae, quae simplicitatis caussa item evanescant, ita, ut sint

$$(15\text{a}) \qquad u_0 = 0 \qquad v_0 = 0.$$

$\frac{dZ_z}{dz} = 0$ i. e. $Z_z =$ constans, quam $= 0$ ponere licet. Quum ex (12) in $Z_z =$ valores quantitatum x_x x_y etc. substituantur, hanc aequationem ad definiendum w_0 obtinemus,

$$\frac{dw_0}{dz} = -\frac{\Theta}{1+\Theta}(z(p_1 - q) + \varepsilon + \varepsilon_1)$$

atque integrando:

$$(15\text{b}) \qquad w_0 = -\frac{z\Theta}{1+\Theta}\left(z\frac{p_1 - q}{2} + \varepsilon + \varepsilon_1\right) + \text{Constans}$$

Pro $z = 0$ $w_0 = 0$ esse necesse est. Ponemus igitur Constantem $= 0$.

In lamina non crystallina ergo invenitur

$$u = z(-qx + py) + \varepsilon x + \sigma y$$

$$(16) \quad v = z(px + p_1 y) + \varepsilon_1 y$$

$$w = -\frac{z\Theta}{1+\Theta}\left(z\frac{(p_1 - q)}{2} + \varepsilon + \varepsilon_1\right) + \frac{qx^2}{2} - pxy - \frac{p_1 y^2}{2}$$

Substitutis quantitatibus x_x x_y etc. in F, haec functio fit

$$(17) \quad F = K\left((-qz + \varepsilon)^2 + (p_1 z + \varepsilon_1)^2 + \frac{\Theta}{1+\Theta}(z(p_1 - q) + \varepsilon + \varepsilon_1)^2 + \frac{(2pz + \sigma)^2}{2}\right)$$

Quum crassitudo laminae $2c$, comparata cum valoribus, quos x et y accipere possunt, sumatur infinite parva, aequatio (3) adhibenda est. Secundum z integratio fiet inter $-c$ et $+c$, secundum x et y super elementum areae mediae $da\,db$ extendatur. Quum F solius z functio sit, obtinetur facile

$$\int F\,dx\,dy\,dz = da\,db\int_{-c}^{+c} F\,dz.$$

Signum $\sum$ in aequatione (3), commutatur in signum $\int$

(18) $\int_{-c}^{+c} F\,dz$ invenitur

$$= \frac{2Kc}{3} \left. \begin{array}{l} \Big\{ c^2\,(q^2 + p_1{}^2 + 2p^2) + 3\left(\varepsilon^2 + \varepsilon_1{}^2 + \frac{\sigma^2}{2}\right) \\ + \frac{\Theta}{1+\Theta}\,(c^2\,(p_1 - q)^2 + 3\,(\varepsilon + \varepsilon_1)^2 \Big\} \end{array} \right\} = f.$$

Lamina porro libere oscillet, quare sumendum est momentum externarum virium $\delta\,\Omega = 0$.

Fit igitur aequatio (3)

$$\delta \int dt\,da\,db\,(T - f) = o,$$

ubi T dimidia vis viva ex coordinatis formanda est. Quibus enim differentiatis secundum t, quoti differentiales in secundam potentatem evehendi et addendi sunt, tum summa per dz multiplicata integranda est usque ad eosdem fines ac $\int F\,dz$.

Ex definitione quantitatum $\varepsilon\ \varepsilon_1$ sequitur $\frac{d\varepsilon}{dt}\ \frac{d\varepsilon_1}{dt}$ non fore ∞ ad $\frac{d\xi}{dt}, \frac{d\eta}{dt}, \frac{d\zeta}{dt}$, nisi $\frac{d\xi}{da}, \frac{d\eta}{da}, \frac{d\zeta}{da}, \frac{d\xi}{db}, \frac{d\eta}{db}, \frac{d\zeta}{db}$, fiant ∞ ad $\xi\ \eta\ \zeta$, quem casum exclusum volumus in tota disquisitione.

Abhinc sequitur ex tribus primis aequationibus (8) $\frac{d\alpha_0}{dt}, \frac{d\beta_0}{dt}, \frac{d\gamma_0}{dt}$ non fore ∞ ad $\frac{d\xi}{dt}\ \frac{d\eta}{dt}\ \frac{d\zeta}{dt}$, ex aliis aequationibus (8) $\frac{d\sigma}{dt}\ \frac{d\alpha_1}{dt}\ \frac{d\beta_1}{dt}\ \frac{d\gamma_1}{dt}$ non fore ∞ ad $\frac{d\xi}{dt}\ \frac{d\eta}{dt}\ \frac{d\zeta}{dt}$ et ratione habita sex relationum inter novem cosinus, quantitates $\frac{d\alpha_2}{dt}\ \frac{d\beta_2}{dt}\ \frac{d\gamma_2}{dt}$ eadem qualitate esse. Sub simili conditione aequationes (9) dicunt, nullam quantitatum $\frac{dp}{dt}\ \frac{dq}{dt}\ \frac{dp_1}{dt}$ fore ∞ ad $\frac{d\alpha_6}{dt}$ etc.

Quare quotos differentiales coordinatarum, neglectis quantitatibus superiorum ordinum scribere licebit

$$\frac{d\xi}{dt}, \qquad \frac{d\eta}{dt}, \qquad \frac{d\zeta}{dt},$$

integratione secundum z facta, invenitur igitur

(19) $$\int da\,db\,T = \rho c \int da\,db\,\left(\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2\right)$$

(ubi ρ laminae densitatem significat).

Quum lamina tantum infinite paullum ex aequilibrii statu progrediatur, de quantitatibus $\alpha_0\,\beta_0\,\gamma_0$ etc. constituatur $\alpha_0\,\beta_1\,\gamma_2$ quantitatibus secundi ordinis distare ab unitate, reliquas $\alpha_1\,\alpha_2\,\beta_0\,\beta_2\,\gamma_0\,\gamma_1$ quantitates esse primi ordinis, ita ut axes $x\,y\,z$ cum $\xi\,\eta\,\zeta$ fere coincidant.

Ex aequationibus (10) posito $x=0$ $y=0$ $z=0$ sequitur porro ε, σ, ε_1 esse quantitates infinite parvas, quia quoti differentiales quantitatum $u\,v\,w$ secundum $x\,y\,z$ sint infinite parvi (vid. loc. cit.) Quare ex aequationibus (8) oriuntur hae:

$$(20)\qquad \begin{aligned} \frac{d\xi}{da} &= \alpha_0+\varepsilon, & \frac{d\xi}{db} &= \sigma+\alpha_1 \\ \frac{d\eta}{da} &= \beta_0, & \frac{d\eta}{db} &= \beta_1+\varepsilon_1 \\ \frac{d\zeta}{da} &= \gamma_0, & \frac{d\zeta}{db} &= \gamma_1 \end{aligned}$$

Relationes sex inter novem cosinus fiunt:

$$(21)\quad \beta_2+\gamma_1=0, \qquad \alpha_2+\gamma_0=0, \qquad \alpha_1+\beta_0=0.$$

Ex aequationibus (9), ope relationum (21) sequuntur, neglectis quantitatibus superiorum ordinum

$$(22)\qquad q=\frac{d\gamma_0}{da}, \qquad -p=q_1=\frac{d\gamma_0}{db}, \qquad p_1=-\frac{d\gamma_1}{db}$$

Conjunctione aequationum (20), (22) obtinentur relationes

$$(23)\qquad q=\frac{d^2\zeta}{da^2}, \qquad p=-\frac{d^2\zeta}{da\,db}, \qquad p_1=-\frac{d^2\zeta}{db^2}$$

Posito igitur ex (18)

$$(24)\qquad f=\frac{2\,Kc^3}{3}f_1+\frac{2\,Kc}{3}f_2$$

fit

$$(24\,\mathrm{a})\quad f_1=\left(\frac{d^2\zeta}{da^2}\right)^2+\left(\frac{d^2\zeta}{db^2}\right)^2+2\left(\frac{d^2\zeta}{da\,db}\right)^2+\frac{\Theta}{1+\Theta}\left(\frac{d^2\zeta}{da^2}+\frac{d^2\zeta}{db^2}\right)^2$$

Quum in expressione f_2 secundae potestates quantitatum ε ε_1 σ solae involvantur, has quantitates tantum ad primum ordinem calculare necesse est, quare scribere licebit

$$(25)\qquad \begin{aligned} \varepsilon &= \frac{d\xi}{da}-1. \\ \varepsilon_1 &= \frac{d\eta}{db}-1. \end{aligned}$$

$$\sigma = \frac{d\xi}{db} + \frac{d\eta}{da},$$

(propter relationem $\beta_0 + \alpha_1 = 0$).

Invenitur itaque

$$(\varepsilon + \varepsilon_1)^2 = \left(\frac{d\xi}{da} + \frac{d\eta}{db}\right)^2 - 4\left(\frac{d\xi}{da} + \frac{d\eta}{db}\right) + 4.$$

$$\varepsilon^2 + \varepsilon_1^2 + \frac{\sigma^2}{2} = \left(\frac{d\xi}{da}\right)^2 + \left(\frac{d\eta}{db}\right)^2 - 2\left(\frac{d\eta}{db} + \frac{d\xi}{da}\right) + 2 + \tfrac{1}{2}\left(\frac{d\xi}{db} + \frac{d\eta}{da}\right)^2.$$

Quibus substitutis in f_2, fit

$$\text{(24b)} \qquad f_2 = \frac{2}{(1+\Theta)} \Big\{ (1+3\Theta) + (1+3\Theta)\left(\frac{d\xi}{da} + \frac{d\eta}{db}\right)$$
$$+ \frac{(1+2\Theta)}{2}\left(\left(\frac{d\xi}{da}\right)^2 + \left(\frac{d\eta}{db}\right)^2\right) + 2\Theta\frac{d\xi}{da}\frac{d\eta}{db} + \frac{(1+\Theta)\left(\frac{d\xi}{db} + \frac{d\eta}{da}\right)^2}{2}\Big\}.$$

Adnotamus f_1 solius ζ, f_2 quantitatum ξ et η esse functiones.

Quum variationes $\delta\xi$, $\delta\eta$, $\delta\zeta$ non dependeant inter se, earum coefficientes singuli $= 0$ ponendi sunt. Habetur itaque, si ρ significat densitatem laminae

$$\text{(26a)} \qquad \delta\iiint dt\, da\, db \left(\rho\left(\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2\right) - \frac{2K}{3} f_2\right) = 0.$$

$$\text{(26b)} \qquad \delta\iiint dt\, da\, db \left(\rho\left(\frac{d\zeta}{dt}\right)^2 - \frac{2Kc^2}{3} f_1\right) = 0.$$

Aequatio (26b) est eadem, quam cl. Kirchhoff in commentatione titulata „Ueber das Gleichgewicht und die Bewegung einer elastischen Scheibe" in diarii Crelliani vol. XL pag. 66 No. 14 communicavit, cujus explicationem vid. in loco citato. Aequationes finales postea commemorabimus. Nunc nobis liceat de prima paullum disserere, quae dat oscillationes longitudinales laminae:

Elementum arcus curvae, quae laminam definit sit ds, angulus ϕ vocetur, qui est inter normalem curvae interiorem et positivum axem x, qui ita volvendus est (ut normali parallelus fiat), ut peracto quadrante cum positivo axi y coincidat. F significante quamlibet functionem coordinatarum orthogonalium a et b facile inveniuntur relationes, a cl. Gauss datae:

$$\text{(27)} \qquad \iint da\, db \frac{dF}{da} = -\int \cos\phi\, ds\, F$$

$$\iint da\, db \frac{dF}{db} = -\int \sin\phi\, ds\, F$$

ubi integratio duplex in aream mediam a curva definitam, simplex tamen in curvam laminam definientem extendenda est.

Quarum aequationum ope fiunt integralia:

$$-\int\!\!\int da\, db\,(1+3\Theta)\left(\frac{d\delta\xi}{da}+\frac{d\delta\eta}{db}\right)=(1+3\Theta)\int ds\,(\delta\xi \cos\phi+\delta\eta \sin\phi),$$

$$\int\!\!\int da\, db\,(1+2\Theta)\left(\frac{d\xi}{da}\frac{d\delta\xi}{da}+\frac{d\eta}{db}\frac{d\delta\eta}{db}\right)=-(1+2\Theta)\int ds\left(\frac{d\xi}{da}\delta\xi\cos\phi+\frac{d\eta}{db}\delta\eta\sin\phi\right)$$
$$-\int\!\!\int da\, db\,(1+2\Theta)\left(\frac{d^2\xi}{da^2}\delta\xi+\frac{d^2\eta}{db^2}\delta\eta\right),$$

$$\int\!\!\int da\, db\,(1+\Theta)\left(\frac{d\xi}{db}+\frac{d\eta}{da}\right)\frac{d\delta\xi}{db}=-(1+\Theta)\int ds\left(\frac{d\xi}{db}+\frac{d\eta}{da}\right)\delta\xi\sin\phi$$
$$-(1+\Theta)\int\!\!\int da\, db\left(\frac{d^2\xi}{db^2}+\frac{d^2\eta}{da\,db}\right)\delta\xi,$$

$$\int\!\!\int da\, db\,(1+\Theta)\left(\frac{d\xi}{db}+\frac{d\eta}{da}\right)\frac{d\delta\eta}{da}=-(1+\Theta)\int ds\left(\frac{d\xi}{db}+\frac{d\eta}{da}\right)\delta\eta\cos\phi$$
$$-\int\!\!\int da\, db\left(\frac{d^2\xi}{da\,db}+\frac{d^2\eta}{da^2}\right)(1+\Theta)\,\delta\eta,$$

$$\Theta\int\!\!\int da\, db\left(\frac{d\xi}{da}\frac{d\delta\eta}{db}+\frac{d\eta}{db}\frac{d\delta\xi}{da}\right)=-\Theta\int ds\left(\frac{d\xi}{da}\delta\eta\sin\phi+\frac{d\eta}{db}\delta\xi\cos\phi\right)$$
$$-\Theta\int\!\!\int da\, db\left(\frac{d^2\xi}{da\,db}\delta\eta+\frac{d^2\eta}{da\,db}\delta\xi\right).$$

Fit igitur, posito $\frac{3}{2}k=\frac{K}{1+\Theta}$

$$\tfrac{1}{2}\delta\int dt\, da\, db\left(\rho\left(\left(\frac{d\xi}{dt}\right)^2+\left(\frac{d\eta}{dt}\right)^2\right)-\frac{2K}{3}f_2\right)=\int\Big[\rho\left(\frac{d\xi}{dt}\frac{d\delta\xi}{dt}+\frac{d\eta}{dt}\frac{d\delta\eta}{dt}\right)$$
$$+k\Big\}\delta\xi\left((1+2\Theta)\frac{d^2\xi}{da^2}+(1+\Theta)\left(\frac{d^2\xi}{db^2}+\frac{d^2\eta}{da\,db}\right)+\Theta\frac{d^2\eta}{da\,db}\right)$$
$$+\delta\eta\left((1+2\Theta)\frac{d^2\eta}{db^2}+(1+\Theta)\left(\frac{d^2\eta}{da^2}+\frac{d^2\xi}{da\,db}\right)+\Theta\frac{d^2\xi}{da\,db}\Big\{\Big]\,dt\, da\, db\right.$$

$$-k\int\!\!\int dt\, ds\Big[\delta\xi\Big\}(1+3\Theta)\cos\phi-(1+2\Theta)\frac{d\xi}{da}\cos\phi-(1+\Theta)\left(\frac{d\xi}{db}+\frac{d\eta}{da}\right)\sin\phi$$
$$-\Theta\frac{d\eta}{db}\cos\phi\Big\{$$
$$+\delta\eta\Big\}(1+3\Theta)\sin\phi-(1+2\Theta)\frac{d\eta}{db}\sin\phi-(1+\Theta)\left(\frac{d\eta}{da}+\frac{d\xi}{db}\right)\cos\phi-\Theta\frac{d\xi}{da}\sin\phi\Big\{\Big].$$

Integratis igitur $\frac{d\delta\xi}{dt}, \frac{d\delta\eta}{dt}$ partim secundum t, ponantur factores variationum $\iint dt\, da\, db\, d\xi$, $\iint dt\, da\, db\, d\eta$, $\iint dt\, ds\, d\xi$, $\iint dt\, ds\, d\eta$ singuli $= 0$, quare obtinentur aequationes differentiales

$$(28)\quad \begin{aligned} \frac{\rho}{k}\frac{d^2\xi}{dt^2} &= (1+2\Theta)\frac{d^2\xi}{da^2} + (1+\Theta)\left(\frac{d^2\xi}{db^2} + \frac{d^2\eta}{da\, db}\right) + \Theta\frac{d^2\eta}{da\, db}. \\ \frac{\rho}{k}\frac{d^2\eta}{dt^2} &= (1+2\Theta)\frac{d^2\eta}{db^2} + (1+\Theta)\left(\frac{d^2\eta}{da^2} + \frac{d^2\xi}{da\, db}\right) + \Theta\frac{d^2\xi}{da\, db}. \end{aligned}$$

Pro finibus laminae i. e. in margine valent

$$(29)\quad \begin{aligned} 0 &= (1+3\Theta)\cos\phi - (1+2\Theta)\frac{d\xi}{da}\cos\phi - (1+\Theta)\left(\frac{d\xi}{db} + \frac{d\eta}{da}\right)\sin\phi - \Theta\frac{d\eta}{db}\cos\phi. \\ 0 &= (1+3\Theta)\sin\phi - (1+2\Theta)\frac{d\eta}{db}\sin\phi - (1+\Theta)\left(\frac{d\xi}{db} + \frac{d\eta}{da}\right)\cos\phi - \Theta\frac{d\xi}{da}\sin\phi. \end{aligned}$$

Eadem ratione aequationem (26b) tractans, cl. Kirchhoff invenit aequationes (loco citato)

$$0 = \rho\frac{d^2\zeta}{dt^2} + \tfrac{2}{3}\frac{1+2\Theta}{1+\Theta}c^2 K\left(\frac{d^4\zeta}{da^2} + 2\frac{d^4\zeta}{da^2 db^2} + \frac{d^4\zeta}{db^2}\right)$$

pro finibus

$$(30)\quad \left\{ \begin{aligned} 0 = &\frac{1+2\Theta}{1+\Theta}\left(\left(\frac{d^3\zeta}{da^3} + \frac{d^3\zeta}{da\, db^2}\right)\cos\phi + \left(\frac{d^3\zeta}{da^2\, db} + \frac{d^3\zeta}{db^3}\right)\sin\phi\right). \\ &- \frac{d}{ds}\left(\frac{d^2\zeta}{da\, db}(\cos^2\phi - \sin^2\phi) + \left(\frac{d^2\zeta}{db^2} + \frac{d^2\zeta}{da^2}\right)\cos\phi\,\sin\phi\right) \\ 0 = &\frac{\Theta}{1+\Theta}\left(\left(\frac{d^2\zeta}{da^2} + \frac{d^2\zeta}{db^2}\right) + \frac{d^2\zeta}{da^2}\cos^2\phi + 2\frac{d^2\zeta}{da\, db}\cos\phi\,\sin\phi + \frac{d^2\zeta}{db^2}\sin^2\phi\right). \end{aligned} \right.$$

Jam transeamus ad laminam crystallinam.

§. 5.

In mediis crystallinis ponendae sunt loco quantitatum X_x X_y etc. functiones lineares quantitatum x_x x_y etc., in quibus coefficientes horizontales sint aequales coefficientibus verticalibus.

Ponamus igitur

$$(31)\quad \begin{aligned} X_x &= a_{00}x_x + a_{01}x_y + a_{11}y_y + a_{02}x_z + a_{12}y_z + a_{22}z_z \\ X_y &= a_{01}x_x + b_{00}x_y + b_{11}y_y + b_{02}x_z + b_{12}y_z + b_{22}z_z \\ Y_y &= a_{11}x_x + b_{11}x_y + c_{11}y_y + c_{02}x_z + c_{12}y_z + c_{22}z_z \end{aligned}$$

$$X_z = a_{02}x_x + b_{02}x_y + c_{02}y_y + d_{02}x_z + d_{12}y_z + d_{22}z_z$$
$$Y_z = a_{12}x_x + b_{12}x_y + c_{12}y_y + d_{12}x_z + e_{12}y_z + e_{22}z_z$$
$$Z_z = a_{22}x_x + b_{22}x_y + c_{22}y_y + d_{22}x_z + e_{22}y_z + f_{22}z_z.$$

Quum habeamus rationem aequationum (13a), inveniuntur, constantibus evanescentibus,

$$(32) \qquad X_z = 0, \qquad Y_z = 0, \qquad Z_z = 0.$$

Fit igitur functio F secundum theorema Eulerianum

$$(33) \qquad 2F = x_x X_x + y_y Y_y + x_y X_y.$$

Posito

$$a_{02}x_x + b_{02}x_y + c_{02}y_y = A_0.$$
$$a_{12}x_x + b_{12}x_y + c_{12}y_y = A_1.$$
$$a_{22}x_x + b_{22}x_y + c_{22}y_y = A_2.$$

facile intelligitur, fore

$$(33\text{a}) \qquad 2F = 2F_1 + x_z A_0 + y_z A_1 + z_z A_2$$

ubi

$$(33\text{b}) \qquad \begin{aligned} 2F_1 &= x_x(a_{00}x_x + a_{01}x_y + a_{11}y_y) \\ &\quad + x_y(a_{01}x_x + b_{00}x_y + b_{11}y_y) \\ &\quad + y_y(a_{11}x_x + b_{11}x_x + c_{11}y_y) \end{aligned}$$

brevitatis caussa positum est.

Habemus porro ex $X_z = 0$, $Y_z = 0$, $Z_z = 0$:

$$d_{02}x_z + d_{12}y_z + d_{22}z_z = -A_0.$$
$$d_{12}x_z + e_{12}y_z + e_{22}z_z = -A_1.$$
$$d_{22}x_z + e_{22}y_z + f_{22}z_z = -A_2.$$

Ponatur determinans

$$\begin{vmatrix} d_{02} & d_{12} & d_{22} \\ d_{12} & e_{12} & e_{22} \\ d_{22} & e_{22} & f_{22} \end{vmatrix} = D.$$

porro

$$(34) \qquad \frac{\partial D}{\partial d_{02}} \frac{1}{D} = D_{02}, \qquad \frac{\partial D}{\partial d_{12}} \frac{1}{D} = D_{12}, \qquad \frac{\partial D}{\partial d_{22}} \frac{1}{D} = D_{22}$$
$$\frac{\partial D}{\partial e_{12}} \frac{1}{D} = E_{12}, \qquad \frac{\partial D}{\partial e_{22}} \frac{1}{D} = E_{22}, \qquad \frac{\partial D}{\partial f_{22}} \frac{1}{D} = F_{22}$$

theoria solutionis aequationum linerarium dat

$$x_z = -(A_0 D_{02} + A_1 D_{12} + A_2 D_{22})$$
$$y_z = -(A_0 D_{12} + A_1 E_{12} + A_2 E_{22})$$
$$z_z = -(A_0 D_{22} + A_1 E_{22} + A_2 F_{22})$$

Fit igitur

$$\begin{aligned} A_0 x_z + A_1 y_z + A_2 z_z = x_x & (a_{02} x_z + a_{12} y_z + a_{22} z_z) \\ + x_y & (b_{02} x_z + b_{12} y_z + b_{22} z_z) \\ + y_y & (c_{02} x_z + c_{12} y_z + c_{22} z_z) \end{aligned}$$

substitutione quantitatum x_z, y_z, z_z.

$A_0 x_z + A_1 y_z + A_2 z_z =$

$$\begin{aligned} & - x_z (A_0 (a_{02} D_{02} + a_{12} D_{12} + a_{22} D_{22}) \\ & \quad + A_1 (a_{02} D_{12} + a_{12} E_{12} + a_{22} E_{22}) \\ & \quad + A_2 (a_{02} D_{22} + a_{12} E_{22} + a_{22} F_{22})) \\ & - x_y (A_0 (b_{02} D_{02} + b_{12} D_{12} + b_{22} D_{22}) \\ & \quad + A_1 (b_{02} D_{12} + b_{12} E_{12} + b_{22} E_{22}) \\ & \quad + A_2 (b_{02} D_{22} + b_{12} E_{22} + b_{22} F_{22})) \\ & - y_y (A_0 (c_{02} D_{02} + c_{12} D_{12} + c_{22} D_{22}) \\ & \quad + A_1 (c_{02} D_{12} + c_{12} E_{12} + c_{22} E_{22}) \\ & \quad + A_2 (c_{02} D_{22} + c_{12} E_{22} + c_{22} F_{22})) \end{aligned}$$

Brevitatis caussa vocentur

$$\left\{ \begin{aligned} a_{02} D_{02} + a_{12} D_{12} + a_{22} D_{22} &= \alpha_a. \\ a_{02} D_{12} + a_{12} E_{12} + a_{22} E_{22} &= \beta_a. \\ a_{02} D_{22} + a_{12} E_{22} + a_{22} F_{22} &= \gamma_a. \end{aligned} \right.$$

$$(35) \qquad \left\{ \begin{aligned} b_{02} D_{02} + b_{12} D_{12} + b_{22} D_{22} &= \alpha_b. \\ b_{02} D_{12} + b_{12} E_{12} + b_{22} E_{22} &= \beta_b. \\ b_{02} D_{22} + b_{12} E_{22} + b_{22} F_{22} &= \gamma_b. \end{aligned} \right.$$

$$\left\{ \begin{aligned} c_{02} D_{02} + c_{12} D_{12} + c_{22} D_{22} &= \alpha_c. \\ c_{02} D_{12} + c_{12} E_{12} + c_{22} E_{22} &= \beta_c. \\ c_{02} D_{22} + c_{22} E_{22} + c_{22} F_{22} &= \gamma_c; \text{ est} \end{aligned} \right.$$

igitur $A_0 x_z + A_1 y_z + A_2 z_z$

$$= - x_x (A_0 \alpha_a + A_1 \beta_a + A_2 \gamma_a) - x_y (A_0 \alpha_b + A_1 \beta_b + A_2 \gamma_b) - y_y (A_0 \alpha_c + A_1 \beta_c + A_2 \gamma_c)$$

Substituantur porro valores quantitatum A_0, A_1, A_2 per x_x, x_y, y_z expressi.

$$A_0 x_z + A_1 y_z + A_2 z_z =$$

$$- \Big\{ + x_x^2 (\alpha_a a_{02} + \beta_a a_{12} + \gamma_a a_{22})$$
$$+ x_y^2 (\alpha_b b_{02} + \beta_b b_{12} + \gamma_b b_{22})$$
$$+ y_y^2 (\alpha_c c_{02} + \beta_c c_{12} + \gamma_c c_{22})$$
$$+ x_x x_y (\alpha_b a_{02} + \beta_b a_{12} + \gamma_b a_{22} + \alpha_a b_{02} + \beta_a b_{12} + \gamma_a b_{22})$$
$$+ x_x y_y (\alpha_c a_{02} + \beta_c a_{12} + \gamma_c a_{22} + \alpha_a c_{02} + \beta_a c_{12} + \gamma_a c_{22})$$
$$+ x_y y_y (\alpha_c b_{02} + \beta_c b_{12} + \gamma_c b_{22} + \alpha_b c_{02} + \beta_b c_{12} + \gamma_b c_{22}) \Big\}$$

$2F_1$ fit $= a_{00} x_x^2 + b_{00} x_y^2 + c_{11} y_y^2 + 2 x_x x_y a_{01} + 2 x_x y_y a_{11} + 2 x_y y_y b_{11}$.

$2F$ igitur hanc induit formam

$$(36) \qquad = (0,0)\, x_x^2 + 2(0,1)\, x_x x_y + 2(0,2)\, x_x y_y + (1,1)\, x_y^2 + 2(1,2)\, x_y y_y + (2,2)\, y_y^2,$$

ubi brevitatis caussa est

$$(0,0) = a_{00} - (\alpha_a a_{02} + \beta_a a_{12} + \gamma_a a_{22})$$
$$(1,1) = b_{00} - (\alpha_b b_{02} + \beta_b b_{12} + \gamma_b b_{22})$$
$$(36\,a) \qquad (2,2) = c_{11} - (\alpha_c c_{02} + \beta_c c_{12} + \gamma_c c_{22})$$
$$2\,(0,1) = 2a_{01} - (\alpha_b a_{02} + \beta_b a_{12} + \gamma_b a_{22} + \alpha_a b_{02} + \beta_a b_{12} + \gamma_a b_{22})$$
$$2\,(0,2) = 2a_{11} - (\alpha_c a_{02} + \beta_c a_{12} + \gamma_c a_{22} + \alpha_a c_{02} + \beta_a c_{12} + \gamma_a c_{22})$$
$$2\,(1,2) = 2b_{11} - (\alpha_c b_{02} + \beta_c b_{12} + \gamma_c b_{22} + \alpha_b c_{02} + \beta_b c_{12} + \gamma_c c_{22}).$$

Etiam nunc formandum est

$\int_{-c}^{+c} F\,dz$, quod componitur ex his integralibus

$$\int_{-c}^{+c} x_x^2\,dz \text{ (ope aeq. 12)} = \int_{-c}^{+c} (-qz + \varepsilon)^2\,dz = \frac{2q^2c^3}{3} + 2\frac{\varepsilon^2 c}{3}.$$

$$\int_{-c}^{+c} x_y^2\,dz = \int_{-c}^{+c} (2pz + \sigma)^2\,dz = \frac{8p^2c^3}{3} + \frac{2\sigma^2 c}{3}.$$

$$\int_{-c}^{+c} y_y^2\,dz = \int_{-c}^{+c} (p_1 z + \varepsilon_1)^2\,dz = \frac{2p_1^2c^3}{3} + \frac{2\varepsilon_1^2 c}{3}.$$

$$\int_{-c}^{+c} x_x x_y\,dz = \int_{-c}^{+c} (-2pqz^2 + z(2p\varepsilon - q\sigma) + \varepsilon\sigma)\,dz = -\frac{4c^3pq}{3} + 2c\varepsilon\sigma.$$

$$\int_{-c}^{+c} x_x y_y\,dz = \int_{-c}^{+c} (-p_1 qz^2 + z(-q\varepsilon_1 + p_1\varepsilon) + \varepsilon\varepsilon_1)\,dz = -\frac{2c^3p_1q}{3} + 2c\,\varepsilon\,\varepsilon_1.$$

$$\int_{-c}^{+c} x_y y_y\,dz = \int_{-c}^{+c} (2pp_1 z^2 + z(2p\varepsilon_1 + p_1\sigma) + \varepsilon_1\sigma)\,dz = \frac{4pp_1c^3}{3} + 2c\sigma\varepsilon_1.$$

Quae integralia multiplicentur resp. per $\frac{(0,0)}{2}$, $\frac{(1,1)}{2}$, $\frac{(2,2)}{2}$, (0,1), (0,2) (1,2) et addantur.

Tum in sinistra parte obtinetur $\int_{-c}^{+c} F\,dz$, in dextra

$$\frac{c^3}{3} f_1 + \frac{c}{3} f_2 = f, \text{ ubi} \tag{37}$$

$$f_1 = (0,0)\, q^2 - 4\,(0,1)\, pq - 2\,(0,2)\, p_1 q + 4\,(1,1)\, p^2 + 4\,(1,2)\, pp_1 + (2,2)\, p_1^2.$$

$$f_2 = (0,0)\, \varepsilon^2 + 2\,(0,1)\, 3\varepsilon\sigma + 2\,(0,2)\, 3\varepsilon\varepsilon_1 + 11\,\sigma^2 + 2\,(1,2)\, 3\sigma\varepsilon_1 + (2,2)\, \varepsilon_1^2.$$

Quum deliberatio, ex qua aequationes (20), (23), (25) deductae sunt, nihil habeat laminae non crystellinae proprium, ea nobis hic etiam-uti licet.

In f_1 et f_2 igitur ponendum est

$$-p = \frac{\partial^2 \zeta}{\partial a\, \partial b}, \quad -p_1 = \frac{\partial^2 \zeta}{\partial b^2}, \quad q = \frac{\partial^2 \zeta}{\partial a^2}.$$

$$\varepsilon = \frac{\partial \xi}{\partial a} - 1, \quad \varepsilon_1 = \frac{\partial \eta}{\partial b} - 1.$$

$$\frac{\partial \xi}{\partial b} + \frac{\partial \eta}{\partial a} = \sigma.$$

In aequatione (3) porro δT formandum est.

Invenimus supra

$$x_z = -(A_0 D_{02} + A_1 D_{12} + A_2 D_{22}).$$

$$y_z = -(A_0 D_{12} + A_1 E_{12} + A_2 E_{22}).$$

$$z_z = -(A_0 D_{22} + A_1 E_{22} + A_2 F_{22}).$$

Quum sit secundum (12)

$$x_z = \frac{\partial u_0}{\partial z}, \quad y_z = \frac{\partial v_0}{\partial z}, \quad z_z = \frac{\partial w_0}{\partial z},$$

positis igitur valoribus quantitatum $x_x\ x_y\ y_y$ ex (12), quae invadunt in $A_0\ A_1\ A_2$, aequationes oriuntur:

$$\frac{\partial u_0}{\partial z} = -\left\{(\varepsilon - qz)\,\alpha_a + (2pz + \sigma)\,\alpha_b + (p_1 z + \varepsilon_1)\,\alpha_c\right\},$$

$$\frac{\partial v_0}{\partial z} = -\left\{(\varepsilon - qz)\,\beta_a + (2pz + \sigma)\,\beta_b + (p_1 z + \varepsilon_1)\,\beta_c\right\},$$

$$\frac{\partial w_0}{\partial z} = -\left\{(\varepsilon - qz)\,\gamma_a + (2pz + \sigma)\,\gamma_b + (p_1 z + \varepsilon_1)\,\gamma_c\right\},$$

atque integrando

$$u_0 = -z\left[\left(\varepsilon - \frac{qz}{2}\right)\alpha_a + (pz+\sigma)\,\alpha_b + \left(\frac{p_1 z}{2} + \varepsilon_1\right)\alpha_c\right],$$

$$(38)\quad v_0 = -z\left[\left(\varepsilon - \frac{qz}{2}\right)\beta_a + (pz+\sigma)\,\beta_b + \left(\frac{p_1 z}{2} + \varepsilon_1\right)\beta_c\right],$$

$$w_0 = -z\left[\left(\varepsilon - \frac{qz}{2}\right)\gamma_a + (pz+\sigma)\,\gamma_b + \left(\frac{p_1 z}{2} + \varepsilon_1\right)\gamma_c\right],$$

Constantes enim propter conditionem, ut u_0 v_0 w_0 pro $z = 0$ evanescant

Qui valores in $u\ v\ w$ (11) substituendi sunt, ut pro laminis crystallinis valeant:

$$u = z\left[-qx + py + \left(\frac{q\alpha_a}{2} - p\alpha_b - p_1\alpha_c\right)z\right] + \varepsilon x + \sigma y - (\alpha_a\varepsilon + \alpha_b\sigma + \alpha_c\varepsilon_1)\,z,$$

$$(39)\quad v = z\left[px + p_1 y + \left(\frac{q\beta_a}{2} - \beta_b p - \frac{\beta_c p_1}{2}\right)z\right] - \varepsilon_1 y - (\beta_a\varepsilon + \beta_c\varepsilon_1)\,z,$$

$$w = \frac{qx^2}{2} - pxy - \frac{\varepsilon_1 y^2}{2} + z^2\left(\frac{q\gamma_a}{2} - p\gamma_b - \frac{\gamma_c p_1}{2}\right) - (\gamma_a\varepsilon + \gamma_b\sigma + \gamma_c\varepsilon_1)\,z$$

Videmus, quum quantitates α_a β_a γ_a etc. sint finitae, qui casus solus tractari potest, formam valorum u, v, w non distare ab illa, quam invenimus in casu laminae non crystallinae, propterea quod easdem ac prius conclusiones facere licebit in formando expressione T, quum coordinatarum valores priores et hi solum in $u\ v\ w$ differunt. Habemus igitur

$$(40)\qquad T = \rho c\left\{\left(\frac{\partial\xi}{\partial t}\right)^2 + \left(\frac{\partial\eta}{\partial t}\right)^2 + \left(\frac{\partial\zeta}{dt}\right)^2.\right\}$$

Quum rursus quantitates $\delta\xi$, $\delta\eta$, $\delta\zeta$ non sint inter se functiones et f_1 solius ζ functio, hae se offerunt relationes

$$(41)\qquad 0 = \partial\int dt\,da\,db\left(\rho\left(\frac{\partial\zeta}{\partial t}\right)^2 - \frac{c^2 f_1}{3}\right).$$

$$(42)\qquad 0 = \delta\int dt\,da\,db\left(\rho\left\{\left(\frac{\partial\xi}{\partial t}\right)^2 + \left(\frac{\partial\eta}{\partial t}\right)^2\right\} - \frac{f_2}{3}\right).$$

$\int da\,db\,\delta f_1$ ex his integralibus componitur:

$$1)\quad \iint da\,db\,\frac{\partial^2\zeta}{\partial a^2}\,\frac{\partial^2\delta\zeta}{\partial a} = \iint da\,db\,\frac{\partial}{\partial a}\left(\frac{\partial^2\zeta}{\partial a^2}\,\frac{\partial\delta\zeta}{\partial a}\right) - \iint da\,db\,\frac{\partial}{\partial a}\left(\frac{\partial^3\zeta}{\partial a^3}\delta\zeta\right)$$
$$+ \iint da\,db\,\frac{\partial^4\zeta}{\partial a^4}\,\delta\zeta.$$

2) $$\iint da\, db \frac{\partial^2 \zeta}{\partial a \partial b} \frac{\partial^2 \delta\zeta}{\partial a \partial b} = \iint da\, db \frac{\partial}{\partial a}\left(\frac{\partial^2 \zeta}{\partial a \partial b} \frac{\partial \delta\zeta}{\partial b}\right) - \iint \frac{\partial}{\partial b}\left(\frac{\partial^3 \zeta}{\partial a^2 \partial b} \delta\zeta\right) da\, db$$
$$+ \iint da\, db \frac{\partial^4 \zeta}{\partial a^2 \partial b^2} \delta\zeta$$

2a) $$\iint da\, db \frac{\partial^2 \zeta}{\partial a \partial b} \frac{\partial^2 \delta\zeta}{\partial a \partial b} = \iint da\, db \frac{\partial}{\partial b}\left(\frac{\partial^2 \zeta}{\partial a \partial b} \frac{\partial \delta\zeta}{\partial a}\right) - \iint \frac{\partial}{\partial a}\left(\frac{\partial^3 \zeta}{\partial a \partial b^2} \delta\zeta\right) da\, db$$
$$+ \iint da\, db\, \delta\zeta \frac{\partial^4 \zeta}{\partial a^2 \partial b^2}$$

3) $$\iint da\, db \frac{\partial^2 \zeta}{\partial b^2} \frac{\partial^2 \delta\zeta}{\partial b^2} = \iint da\, db \frac{\partial}{\partial b}\left(\frac{\partial^2 \zeta}{\partial b^2} \frac{\partial \delta\zeta}{\partial b}\right) - \iint da\, db \frac{\partial}{\partial b}\left(\frac{\partial^3 \zeta}{\partial b^3} \delta\zeta\right)$$
$$+ \iint da\, db \frac{\partial^4 \zeta}{\partial b^4} \delta\zeta.$$

4) $$\iint da\, db \frac{\partial^2 \zeta}{\partial a^2} \frac{\partial^2 \delta\zeta}{\partial a\, \partial b} = \iint da\, db \frac{\partial}{\partial a}\left(\frac{\partial^2 \zeta}{\partial a^2} \frac{\partial \delta\zeta}{\partial b}\right) - \iint da\, db \frac{\partial \left(\frac{\partial^3 \zeta}{\partial a^3} \delta\zeta\right)}{\partial b}$$
$$+ \iint da\, db \frac{\partial^4 \zeta}{\partial a^3 \partial b} \delta\zeta.$$

5) $$\iint da\, db \frac{\partial^2 \zeta}{\partial b^2} \frac{\partial^2 \delta\zeta}{\partial a\, \partial b} = \iint da\, db \frac{\partial}{\partial b}\left(\frac{\partial^2 \zeta}{\partial b^2} \frac{\partial \delta\zeta}{\partial a}\right) - \iint da\, db \frac{\partial}{\partial a}\left(\frac{\partial^3 \zeta}{\partial b^3} \delta\zeta\right)$$
$$+ \iint da\, db \frac{\partial^4 \zeta}{\partial b^3 \partial a} \delta\zeta.$$

6) $$\iint da\, db \frac{\partial^2 \zeta}{\partial a \partial b} \frac{\partial^2 \delta\zeta}{\partial a^2} = \iint da\, db \frac{\partial}{\partial a}\left(\frac{\partial^2 \zeta}{\partial a\, \partial b} \frac{\partial \delta\zeta}{\partial a}\right) - \iint da\, db \frac{\partial}{\partial a}\left(\frac{\partial^3 \zeta}{\partial a^2\, \partial b} \delta\zeta\right)$$
$$+ \iint da\, db \frac{\partial^4 \zeta}{\partial a^3 \partial b} \delta\zeta.$$

7) $$\iint da\, db \frac{\partial^2 \zeta}{\partial a\, \partial b} \frac{\partial^2 \delta\zeta}{\partial b^2} = \iint da\, db \frac{\partial}{\partial b}\left(\frac{\partial^2 \zeta}{\partial a\, \partial b} \frac{\partial \delta\zeta}{\partial b}\right) - \iint da\, db \frac{\partial}{\partial b}\left(\frac{\partial^3 \zeta}{\partial a\, \partial b^2} \delta\zeta\right)$$
$$+ \iint \frac{\partial^4 \zeta}{\partial a\, \partial b^3} \delta\zeta\, da\, db.$$

8) $$\iint da\, db \frac{\partial^2 \zeta}{\partial a^2} \frac{\partial^2 \delta\zeta}{\partial b^2} = \iint da\, db \frac{\partial}{\partial b}\left(\frac{\partial^2 \zeta}{\partial a^2} \frac{\partial \delta\zeta}{\partial b}\right) - \iint da\, db \frac{\partial}{\partial b}\left(\frac{\partial^3 \zeta}{\partial a^2\, \partial b} \delta\zeta\right)$$
$$+ \iint da\, db \frac{\partial^4 \zeta}{\partial a^2 \partial b^2} \delta\zeta.$$

9) $$\iint da\, db \frac{\partial^2 \zeta}{\partial b^2} \frac{\partial^2 \delta\zeta}{\partial a^2} = \iint da\, db \frac{\partial}{\partial a}\left(\frac{\partial^2 \zeta}{\partial b^2} \frac{\partial \delta\zeta}{\partial a}\right) - \iint da\, db \frac{\partial}{\partial a}\left(\frac{\partial^3 \zeta}{\partial a\, \partial b^2} \delta\zeta\right)$$
$$+ \iint da\, db\, \delta\zeta \frac{\partial^4 \zeta}{\partial a^2\, \partial b^2}.$$

Adhibitis relationibus jam supra propositis

$$\iint da\, db\, \frac{\partial F}{\partial a} = -\int ds \cos\phi\, F.$$

$$\iint da\, db\, \frac{\partial F}{\partial b} = -\int ds \sin\phi\, F, \text{ patet}$$

tres classes integralium reperiri, nempe formam habentes

$$\iint da\, db\, \delta\zeta, \quad \int ds\, \delta\zeta, \quad \int ds \frac{\partial \delta\zeta}{\partial b} \text{ resp.} \int ds \frac{\partial \delta\zeta}{\partial a},$$

quae postrema integralia transformatione quadam formas induent $\int ds \frac{\partial \delta\zeta}{\partial N}$ et $\int ds\, \delta\zeta$ ubi N designat normalem curvae laminam definientis; secundum regulas calculi variationum, integralia singula $= 0$ ponenda sunt.

Quibus deliberatis ponatur

$$\iint \tfrac{1}{2}\, \delta f_1 = \delta\psi_1 + \delta\psi_2 + \delta\psi_3, \text{ ubi}$$

$$\delta\psi_1 = \iint da\, db\, \delta\zeta \Big((0,0) \frac{\partial^4 \zeta}{\partial a^4} + 2\,(0,1) \frac{\partial^4 \zeta}{\partial a^3 \partial b} + 2\,(0,2) \frac{\partial^4 \zeta}{\partial a^2 \partial b^2} + 4\,(1,1) \frac{\partial^4 \zeta}{\partial a^2 \partial b^2}$$
$$+ 2\,(1,2) \frac{\partial^4 \zeta}{\partial a \partial b^3} + (2,2) \frac{\partial^4 \zeta}{\partial b^4} \Big)$$

$$\delta\psi_2 = \int ds\, \delta\zeta \Big\{ (0,0) \frac{\partial^3 \zeta}{\partial a^3} \cos\phi + 2\,(0,1) \Big(\frac{\partial^3 \zeta}{\partial a^3} \sin\phi + \frac{\partial^3 \zeta}{\partial a^2 \partial b} \cos\phi \Big) + (0,2) \Big(\frac{\partial^3 \zeta}{\partial a^2 \partial b} \sin\phi$$
$$+ \frac{\partial^3 \zeta}{\partial a \partial b^2} \cos\phi \Big) + (1,1)\, 2 \Big(\frac{\partial^3 \zeta}{\partial a^2 \partial b} \sin\phi + \frac{\partial^3 \zeta}{\partial a \partial b^2} \cos\phi \Big) + 2\,(1,2) \Big(\frac{\partial^3 \zeta}{\partial b^3} \cos\phi$$
$$+ \frac{\partial^3 \zeta}{\partial a \partial b^2} \sin\phi \Big) + (2,2) \frac{\partial^3 \zeta}{\partial b^3} \sin\phi \Big\}.$$

$$\delta\psi_3 = -\int ds \Big\{ (0,0) \frac{\partial^2 \zeta}{\partial a^2} \frac{\partial \delta\zeta}{\partial a} \cos\phi + 2\,(0,1) \cos\phi \Big\{ \frac{\partial^2 \zeta}{\partial a^2} \frac{\partial \delta\zeta}{\partial b} + \frac{\partial^2 \zeta}{\partial a \partial b} \frac{\partial \delta\zeta}{da} \Big\}$$
$$+ (0,2) \Big(\frac{\partial^2 \zeta}{\partial a^2} \frac{\partial \delta\zeta}{\partial b} \sin\phi + \frac{\partial^2 \zeta}{\partial b^2} \frac{\partial \delta\zeta}{\partial a} \cos\phi \Big) + 2\,(1,1) \frac{\partial^2 \zeta}{\partial a \partial b} \Big(\frac{\partial \delta\zeta}{db} \cos\phi + \frac{\partial \delta\zeta}{\partial a} \sin\phi \Big)$$
$$+ 2\,(1,2) \sin\phi \Big(\frac{\partial^2 \zeta}{\partial b^2} \frac{\partial \delta\zeta}{\partial a} + \frac{\partial^2 \zeta}{\partial a \partial b} \frac{\partial \delta\zeta}{\partial b} \Big) + (2,2) \frac{\partial^2 \zeta}{\partial b^2} \frac{\partial \delta\zeta}{\partial b} \sin\phi \Big\}.$$

Introductis quotis differentialibus secundum normalem N, qui satisfaciunt aequationibus

$$\frac{\partial \delta\zeta}{\partial a} = \frac{\partial \delta\zeta}{\partial N} \cos\phi + \frac{\partial \delta\zeta}{\partial s} \sin\phi$$

$$\frac{\partial \delta\zeta}{\partial b} = \frac{\partial \delta\zeta}{\partial N} \sin\phi - \frac{\partial \delta\zeta}{\partial s} \cos\phi,$$

$\partial\psi_3$ igitur mutatur in:

$$= -\int ds \frac{\partial\delta\zeta}{\partial N}\Big[(0,0)\frac{\partial^2\zeta}{\partial a^2}\cos^2\phi + 2(0,1)\cos\phi\,(\frac{\partial^2\zeta}{\partial a^2}\sin\phi + \frac{\partial^2\zeta}{\partial a\partial b}\cos\phi)$$

$$+ (0,2)(\frac{\partial^2\zeta}{\partial a^2}\sin^2\phi + \frac{\partial^2\zeta}{\partial b^2}\cos^2\phi) + 4(1,1)\frac{\partial^2\zeta}{\partial a\partial b}\sin\phi\cos\phi + 2(1,2)\sin\phi\,(\frac{\partial^2\zeta}{\partial b^2}\cos\phi$$

$$+ \frac{\partial^2\zeta}{\partial a\partial b}\sin\phi) + (2,2)\frac{\partial^2\zeta}{\partial b^2}\sin^2\phi\Big] - \int ds\frac{\partial\delta\zeta}{\partial s}\Big[(0,0)\frac{\partial^2\zeta}{\partial a^2}\sin\phi\cos\phi + 2(0,1)\cos\phi\,\times$$

$$(\frac{\partial^2\zeta}{\partial a\partial b}\sin\phi - \frac{\partial^2\zeta}{\partial a^2}\cos\phi) + (0,2)(\frac{\partial^2\zeta}{\partial a^2} + \frac{\partial^2\zeta}{\partial b^2})\sin\phi\cos\phi + 2(1,1)\frac{\partial^2\zeta}{\partial a\partial b}(\sin^2\phi - \cos^2\phi)$$

$$+ 2(1,2)\sin\phi\,(\frac{\partial^2\zeta}{\partial b^2}\sin\phi - \frac{\partial^2\zeta}{\partial a^2}\cos\phi) - (2,2)\frac{\partial^2\zeta}{\partial b^2}\sin\phi\cos\phi\Big].$$

Quum integratio secundum s in totam curvam extendenda sit, i. e. ad eundem valorem quantitatis s, a quo integrare coepimus, facile intelligitur esse

$$-\int ds\frac{\partial\delta\zeta}{\partial s}F = \int \partial s\,\delta\zeta\frac{\partial F}{\partial s}.$$

Cujus aequationis ratione habita invenitur $\delta\psi_2 + \delta\psi_3 =$

$$\int \partial s\,\delta\zeta\Big[(0,0)\frac{\partial^3\zeta}{\partial a^3}\cos\phi + 2(0,1)(\frac{\partial^3\zeta}{\partial a^3}\sin\phi + \frac{\partial^3\zeta}{\partial a^2\partial b}\cos\phi) + (0,2)(\frac{\partial^3\zeta}{\partial a^2\partial b}\sin\phi$$

$$+ \frac{\partial^3\zeta}{\partial a\partial b^2}\cos\phi) + 2(1,1)(\frac{\partial^3\zeta}{\partial a^2\partial b}\sin\phi + \frac{\partial^3\zeta}{\partial a\partial b^2}\cos\phi) + 2(1,2)(\frac{\partial^3\zeta}{\partial b^3}\cos\phi + \frac{\partial^3\zeta}{\partial a\partial b^2}\sin\phi)$$

$$+ (2,2)\frac{\partial^3\zeta}{\partial b^3}\sin\phi + \frac{\partial}{\partial s}\Big\{(0,0)\frac{\partial^2\zeta}{\partial a^2}\sin\phi\cos\phi + 2(0,1)\cos\phi\,(\frac{\partial^2\zeta}{\partial a\partial b}\sin\phi - \frac{\partial^2\zeta}{\partial a^2}\cos\phi)$$

$$+ (0,2)(\frac{\partial^2\zeta}{\partial a^2} + \frac{\partial^2\zeta}{\partial b^2})\sin\phi\cos\phi + 2(1,1)\frac{\partial^2\zeta}{\partial a\partial b}(\sin^2\phi - \cos^2\phi) + 2(1,2)\sin\phi\,\times$$

$$\Big\{\frac{\partial^2\zeta}{\partial b^2}\sin\phi - \frac{\partial^2\zeta}{\partial a\partial b}\cos\phi\Big\} - (2,2)\frac{\partial^2\zeta}{\partial b^2}\sin\phi\cos\phi\Big\}\Big] - \int ds\frac{\partial\delta\zeta}{\partial N}\Big\{(0,0)\frac{\partial^2\zeta}{\partial a^2}\cos^2\phi$$

$$+ 2(0,1)\cos\phi\,(\frac{\partial^2\zeta}{\partial a^2}\sin\phi + \frac{\partial^2\zeta}{\partial a\partial b}\cos\phi) + (0,2)(\frac{\partial^2\zeta}{\partial a^2}\sin^2\phi + \frac{\partial^2\zeta}{\partial b^2}\cos^2\phi)$$

$$+ 4(1,1)\frac{\partial^2\zeta}{\partial a\partial b}\sin\phi\cos\phi + 2(1,2)(\frac{\partial^2\zeta}{\partial b^2}\cos\phi + \frac{\partial^2\zeta}{\partial a\partial b}\sin\phi) + (2,2)\frac{\partial^2\zeta}{\partial b^2}\sin^2\phi\Big\}.$$

Jam ad evolvendam functionem f_2 transeamus.

f_2 erat $= (0,0)\,\varepsilon^2 + 2(0,1)\,3\varepsilon\,\sigma + 2(0,2)\,3\varepsilon\,\varepsilon_1 + (1,1)\,\sigma^2 + 2(1,2)\,\varepsilon_1\sigma + (2,2)\,\varepsilon_1^2.$

substitutione quantitatum

$$\varepsilon = \frac{\partial \xi}{\partial a} - 1, \qquad \varepsilon_1 = \frac{\partial \eta}{\partial a} - 1, \qquad \sigma = \frac{\partial \xi}{\partial b} + \frac{\partial \eta}{\partial a} \text{ facta, } f_2 = \phi_0 + \phi_1 + \phi_2$$

ponatur, ubi $\phi_0 = (0,0) + 6(0,2) + (2,2)$.

$$\phi_1 = -2\Big[(0,0)\frac{\partial \xi}{\partial a} + 3(0,1)\left(\frac{\partial \xi}{\partial b} + \frac{\partial \eta}{\partial a}\right) + 3(0,2)\left(\frac{\partial \xi}{\partial a} + \frac{\partial \eta}{\partial b}\right) + 3(1,2)\left(\frac{\partial \xi}{\partial b} + \frac{\partial \eta}{\partial a}\right)$$
$$+ (2,2)\frac{\partial \eta}{\partial b}\Big]$$

$$\phi_2 = (0,0)\left(\frac{\partial \xi}{\partial a}\right)^2 + 2(0,1)3\frac{\partial \xi}{\partial a}\left(\frac{\partial \xi}{\partial b} + \frac{\partial \eta}{\partial a}\right) + 2\cdot 3(0,2)\frac{\partial \xi}{\partial a}\frac{\partial \eta}{\partial b} + (1,1)\left(\frac{\partial \xi}{\partial b} + \frac{\partial \eta}{\partial a}\right)^2$$
$$+ 2(1,2)3\frac{\partial \eta}{\partial b}\left(\frac{\partial \xi}{\partial b} + \frac{\partial \eta}{\partial a}\right) + (2,2)\left(\frac{\partial \eta}{\partial b}\right)^2$$

itaque fit

$$\iint da\, db\, df_2 = \iint da\, db\, (\delta\phi_0 + \delta\phi_1 + \delta\phi_2)$$

$$\iint da\, db\, \delta\phi_0 = 0, \text{ quod } \phi_0 = \text{Constans.}$$

$$\iint da\, db\, \delta\phi_1 = 2\int ds\, \delta\xi\, \{\cos\phi\,(0,0 + 3\,(0,2)) + 3\sin\phi\,((0,1) + (1,2))\}$$
$$+ 2\int ds\, \delta\eta\, [3\cos\phi\,((0,1) + (1,2)) + \sin\phi\,((2,2) + 3(0,2)\}]$$

$$\iint \delta\phi_2\, da\, db = -2\int ds \Big\{(0,0)\frac{\partial \xi}{\partial a}\cos\phi + 3\,(0,1)\left(\left(\frac{\partial \xi}{\partial b} + \frac{\partial \eta}{\partial a}\right)\cos\phi + \frac{\partial \xi}{\partial b}\sin\phi\right)$$
$$+ 3\,(0,2)\frac{\partial \eta}{\partial b}\cos\phi + (1,1)\left(\frac{\partial \xi}{\partial b} + \frac{\partial \eta}{\partial a}\right)\sin\phi + 3\,(1,2)\frac{\partial \eta}{\partial b}\sin\phi\Big\}\,\delta\xi$$
$$-2\int ds\Big\{3\,(0,1)\frac{\partial \xi}{\partial a}\cos\phi + 3\,(0,2)\frac{\partial \xi}{\partial a}\sin\phi + (1,1)\left(\frac{\partial \xi}{\partial b} + \frac{\partial \eta}{\partial a}\right)\cos\phi + 3\,(1,2)\Big(\Big(\frac{\partial \xi}{\partial b}$$
$$+ \frac{\partial \eta}{\partial a}\Big)\sin\phi + \frac{\partial \eta}{\partial b}\cos\phi\Big) + (2,2)\frac{\partial \eta}{\partial b}\sin\phi\Big\}\,\delta\eta - 2\iint da\, db\Big\{(0,0)\frac{\partial^2 \xi}{\partial a^2} + 3\,(0,1)\Big(\frac{\partial^2 \xi}{\partial a\, \partial b}$$
$$+ \frac{\partial}{\partial a}\left(\frac{\partial \xi}{\partial b} + \frac{\partial \eta}{\partial a}\right)\Big) + 3\,(0,2)\frac{\partial^2 \eta}{\partial a\, \partial b} + (1,1)\frac{\partial}{\partial b}\left(\frac{\partial \xi}{\partial b} + \frac{\partial \eta}{\partial a}\right) + 3\,(1,2)\frac{\partial^2 \eta}{\partial b^2}\Big\}\,\delta\xi$$
$$-2\iint da\, db\Big\{3\,(0,1)\frac{\partial^2 \xi}{\partial a^2} + 3\,(0,2)\frac{\partial^2 \xi}{\partial a\, \partial b} + (1,1)\left(\frac{\partial}{\partial a}\left(\frac{\partial \xi}{\partial b} + \frac{\partial \eta}{\partial a}\right) + \frac{\partial^2 \eta}{\partial a\, \partial b}\right)$$
$$+ (2,2)\frac{\partial^2 \eta}{\partial b^2}\Big\}\,\delta\xi.$$

Secundum principia calculi variationum, coefficientes quantitatum $\int\int da\, db\, \delta\xi$, $\int\int da\, db\, \delta\eta$, $\int\int da\, db\, \delta\zeta$ singuli $= 0$ ponendi sunt, item coefficientes integralium, quae extendenda sunt ad curvam, quae laminam definit, $\int ds\, \delta\xi$, $\int ds\, \delta\eta$, $\int ds\, \delta\zeta$ et $\int ds \frac{\partial \delta\zeta}{\partial N}$ evanescere necesse est. Quae posteriores aequationes sunt „conditiones marginis."

Oscillationes transversales itaque his definiuntur aequationibus (facta variatione quantitatis T)

$$(43) \qquad 0 = \frac{3\rho}{c^2}\frac{\partial^2\zeta}{\partial t^2} + (0,0)\frac{\partial^4\zeta}{\partial a^4} + 2\,(0,1)\frac{\partial^4\zeta}{\partial a^3\,\partial b} + 2\,(0,2)\frac{\partial^4\zeta}{\partial a^2\,\partial b^2}$$
$$+ 4\,(1,1)\frac{\partial^4\zeta}{\partial a^2\,\partial b^2} + 2\,(1,2)\frac{\partial^4\zeta}{\partial a\,\partial b^3} + (2,2)\frac{\partial^4\zeta}{\partial b^4}.$$

Pro finibus valent:

$$0 = (0,0)\frac{\partial^3\zeta}{\partial a^3}\cos\phi + 2\,(0,1)\left(\frac{\partial^3\zeta}{\partial a^3}\sin\phi + \frac{\partial^3\zeta}{\partial a^2\,\partial b}\right) + (0,2)\left(\frac{\partial^3\zeta}{\partial a^2\,\partial b}\sin\phi\right.$$
$$\left. + \frac{\partial^3\zeta}{\partial a\,\partial b^2}\cos\phi\right) + 2\,(1,2)\left(\frac{\partial^3\zeta}{\partial a^2\,\partial b}\sin\phi + \frac{\partial^3\zeta}{\partial a\,\partial b^2}\cos\phi + 2\,(1,2)\left(\frac{\partial^3\zeta}{\partial b^3}\cos\phi\right.\right.$$
$$(44) \qquad \left.\left. + \frac{\partial^3\zeta}{\partial a\,\partial b^2}\sin\phi\right) + (2,2)\frac{\partial^3\zeta}{\partial b^2}\sin\phi + \frac{\partial}{\partial s}\left\{(0,0)\frac{\partial^2\zeta}{\partial a^2}\sin\phi\cos\phi\right.$$
$$+ 2\,(0,1)\cos\phi\left\{\frac{\partial^2\zeta}{\partial a\,\partial b}\sin\phi - \frac{\partial^2\zeta}{\partial b^2}\cos\phi\right) + (0,2)\left(\frac{\partial^2\zeta}{\partial a^2} + \frac{\partial^2\zeta}{\partial b^2}\right)\sin\phi\cos\phi$$
$$\left. + 2\,(1,1)\frac{\partial^2\zeta}{\partial a\,\partial b}(\sin^2\phi - \cos^2\phi) + 2\,(1,2)\sin\phi\left(\frac{\partial^2\zeta}{\partial b^2}\sin\phi - \frac{\partial^2\zeta}{\partial a\,\partial b}\cos\phi\right)\right\}$$
$$- (2,2)\frac{\partial^2\zeta}{\partial b^2}\sin\phi\cos\phi.$$

$$0 = (0,0)\frac{\partial^2\zeta}{\partial a^2}\cos^2\phi + 2\,(0,1)\cos\phi\left(\frac{\partial^2\zeta}{\partial a^2}\sin\phi + \frac{\partial^2\zeta}{\partial a\,\partial b}\cos\phi\right) - (0,2)\left(\frac{\partial^2\zeta}{\partial a^2}\sin^2\phi + \right.$$
$$(45) \qquad \left.\frac{\partial^2\zeta}{\partial b^2}\cos^2\phi\right) + 4\,(1,1)\frac{\partial^2\zeta}{\partial a\,\partial b}\sin\phi\cos\phi + 2\,(1,2)\sin\phi\left(\frac{\partial^2\zeta}{\partial b^2}\cos\phi + \frac{\partial^2\zeta}{\partial a\,\partial b}\sin\phi\right)$$
$$+ (2,2)\frac{\partial^2\zeta}{\partial b^2}\sin^2\phi.$$

Oscillationes longitudinales his aequationibus definiuntur

$$(46)\qquad (0,0)\frac{\partial^2\xi}{\partial a^2}+3\,(0{,}1)\Big(\frac{\partial^2\xi}{\partial a\,\partial b}+\frac{\partial}{\partial a}\Big(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\Big)\Big)+3\,(0{,}2)\frac{\partial^2\eta}{\partial a\,\partial b}$$

$$+(1,1)\frac{\partial}{\partial b}\Big(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\Big)+3\,(1{,}2)\frac{\partial^2\eta}{\partial b^2}=3\rho\frac{\partial^2\xi}{\partial t^2}.$$

$$(47)\quad 3\,(0{,}1)\frac{\partial^2\xi}{\partial a^2}+3\,(0{,}2)\frac{\partial^2\xi}{\partial a\,\partial b}+(1{,}1)\Big(\frac{\partial}{\partial a}\Big(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\Big)+\frac{\partial^2\eta}{\partial a\,\partial b}\Big)+(2{,}2)\frac{\partial^2\eta}{\partial b^2}$$

$$=3\rho\frac{\partial^2\eta}{\partial t^2}\ \text{pro finibus:}$$

$$(48)\quad 0=(0,0)\frac{\partial\xi}{\partial a}\cos\phi+3\,(0{,}1)\Big(\Big(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\Big)\cos\phi+\frac{\partial\xi}{\partial b}\sin\phi\Big)+3\,(0{,}2)\frac{\partial\eta}{\partial b}\cos\phi$$

$$+(1,1)\Big(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\Big)\sin\phi+3\,(1{,}2)\frac{\partial\eta}{\partial b}\sin\phi-\cos\phi\,((0{,}0)+3\,(0{,}2))-3\sin\phi((0{,}1)+(1{,}2)).$$

$$(49)\quad 0=3\,(0{,}1)\frac{\partial\xi}{\partial a}\cos\phi+3\,(0{,}2)\frac{\partial\xi}{\partial a}\sin\phi+(1{,}1)\Big(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\Big)\cos\phi$$

$$+3\,(1{,}2)\Big(\Big(\frac{\partial\xi}{\partial b}+\frac{\partial\eta}{\partial a}\Big)\sin\phi+\frac{\partial\eta}{\partial b}\cos\phi\Big)+(2{,}2)\frac{\partial\eta}{\partial b}\sin\phi-3\cos\phi\,((0{,}1)+(1{,}2))$$

$$-\sin\phi\,((2{,}2)+3\,(0{,}2)).$$

Quae aequationes in eas transeunt, quas pro lamina non crystallina invenimus, quum ex (31) et (14) evolvuntur coefficientim comparatione

$$a_{00}=2K(1+\Theta),\ b_{00}=K,\qquad c_{11}=2K(1+\Theta)$$
$$a_{11}=2K\Theta,\qquad a_{22}=2K\Theta,\ c_{22}=2K\Theta.$$
$$d_{02}=K,\qquad e_{12}=K,\quad f_{22}=2K(1+\Theta),$$

ceteris evanescentibus. Ex quibus sequitur

$$D_{02}=\frac{1}{K},\ \ E_{12}=\frac{1}{K},\ \ F_{22}=\frac{1}{2K}(1+\Theta)$$

$$D_{12}=0,\ \ D_{22}=0,\ \ E_{22}=0.$$

igitur

$$\alpha_a=0,\qquad \alpha_b=0,\ \alpha_c=0.$$
$$\beta_a=0,\qquad \beta_b=0,\ \beta_c=0.$$
$$\gamma_a=\frac{\Theta}{1+\Theta},\ \gamma_b=0,\ \gamma_c=\frac{\Theta}{1+\Theta}$$

$$(0{,}0)=\frac{2K(1+2\,\Theta)}{1+\Theta},\qquad 4(1{,}1)+2\,(0{,}2)=\frac{4K(1+2\,\Theta)}{1+\Theta},\qquad (2{,}2)=\frac{2K(1+2\,\Theta)}{1+\Theta},$$

$$(0{,}1)=0,\ (1{,}2)=0$$

Qui valores quum substituantur in aequationes (43) et sequentes, facile intelligitur, eas cum (28) (29) (30) congruere; quae hac ratione iterum sunt demonstratae. Integratio aequationum (43) et sequentium tam difficilis videtur esse, ut in hodierno scientiae analyticae statu fieri non possit.

Necesse erit casum quendam specialem, in quo nonnullae quantitatum (0,0) (0,1) etc. evanescant aut simpliciter inter se dependeant, tractare.

Sed non minus curva laminam definiens respicienda est. In casu circuli aequationes (30) a cl. Kirchhoff integratae sunt, in casu ellipsis nondum contigit integratio. —

VITA.

Natus sum Franciscus Eduardus Gehring, catholicae fidei addictus, Nordhusae die septimo mensis Decembris anni MDCCCXXXVIII, patre Antonio et matre Sophia, e gente Dittmar, quibus adhuc viventibus gaudeo. Quum quinque annos natus essem, parentes mutaverunt domicilium suum et se mecum Regiomontum, quod in Borussia vocatur contulerunt. Ubi primis literarum elementis imbutus discipulis Collegii Fridericiani a rectore ill. F. A. Gotthold adscriptus sum, quod nunc auspiciis ill. Horkel floret. Postea Berolini gymnasium Fridericum Guilelmum frequentavi, cui adhuc rector praeest cl. Ranke. Ubi in matheseos fundamenta me introduxit cl. Schellbach. Vere anni MDCCCLVI testimonium maturitatis adeptus a rectore magnifico cl. Ehrenberg in numerum civium almae literarum universitatis Fridericae Guilelmae adscriptus sum. Nec dubitavi matheseos studio me tribuere. Tum per quinque semestria civis universitatis Albertinae Regiomontanae fui; postea valetudine pectoris et cordis impeditus, studia per semestre intermittere necesse erat, quae tamen Heidelbergae revocavi. Docuerunt me Berolini: cl. et exp. Kummer, Encke, Dove, Arndt; Regiomonti cl. Richelot, Neumann, Luther. Heidelbergae: cl. Hesse, Kirchhoff, Bunsen. Vere hujus anni iterum in numerum civium universitatis Frid. Guilelmae receptus a rect. magnifico cl. Boekh et viro spectabili decano cl. Trendelenburg apud philosophorum ordinem nomen professus sum. Quo ex tempore intersum scholis horum virorum celeb.: Trendelenburg, Kummer, Weierstrass. Quibus omnibus viris optime de me meritis gratias ago semperque agam.

THESES.

1. In rebus mathematicis non exstat methodus omnium optima, sed ab uniuscujusque problematis tractandi natura dependet.
2. Solutione problematis mechanici, semper intuitio geometrica amplificatur, sed rarius cognitio viam contrariam sequitur.
3. Elasticitatis problemata accurate tractari non possunt, nisi adhibito calculo variationum.
4. Criterium generale convergentiae serierum a clarissimo Abel propositum maximi momenti est in hujus rei theoria; nec severitate ab ullo alio superatur.
5. Aequationes lineares differentiales, quarum coefficientes sunt functiones variabilis independentis x, usque huc integratae, ex aequationibus differentialibus, quarum coefficientes sunt constantes, substitutione ortae sunt.
